

One-dimensional fermionic gases with attractive p -wave interaction in a hard-wall trap

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(Dated: Original August 27, 2007; Revised February 1, 2008)

We investigate the ground state of the one-dimensional fermionic system enclosed in a hard-wall trap with attractive contact p -wave interactions. Based on the Bethe ansatz method, the explicit wave function is derived by numerically solving the Bethe ansatz equations for the full physical regimes ($-\infty \leq c_F \leq 0$). With the exact wave function some quantities which are important in many-body physics are obtained, including the one-body density matrix and the momentum distribution of the ground state for finite system. It is shown that the shell structure of the density profiles disappears with the increase of the interaction and in the fermionic Tonks-Girardeau (FTG) limit the density distribution shows the same behavior as that of an ideal Bose gas. However the one-body density matrix and the momentum distribution exhibit completely different structures compared with their bosonic counterparts.

PACS numbers: 03.75.Hh, 05.30.Fk, 05.30.Jp, 67.40.Db

I. INTRODUCTION

Experimental realization of trapped one-dimensional (1D) cold atom systems [1, 2, 3, 4] are triggering more and more theoretical efforts to study the 1D many-body physics beyond the mean-field theory. For the ultracold quantum gases tightly confined in waveguides, the dynamics are effectively described by a 1D model due to the radial degrees of freedom are frozen [5, 6]. Further, the ability of tuning the effective 1D interactions by Feshbach resonance allows experimental access to the very strongly interacting regime where correlation effects are greatly enhanced [7, 8, 9, 10, 11]. In the limit of the Tonks-Girardeau (TG) [12] gas with effective coupling constant $g_{1D} \rightarrow \infty$, the many-body problem of a TG gas can be mapped to that of a free Fermi gas by the Bose-Fermi mapping, which has been verified by two experimental groups [3, 4]. This Bose-Fermi duality was generalized to show the equivalence between a 1D fermionic system and a bosonic one with the reversed role of strong and weak couplings [13]. Recently, the exact ground state of the fermionic TG (FTG) gas, defined as a 1D spin-polarized fermionic gas with infinitely strong attractive p -wave interactions, has been determined by inversely Fermi-Bose mapping to the ideal Bose gas [14, 15, 16, 17].

A key experimental challenge is to obtain superfluidity with pairs in nonzero orbital angular momentum states by using p -wave, or maybe even d -wave Feshbach resonances. In general, the p -wave interaction is very weak comparing with the s -wave interaction. However, for a spin-polarized fermionic gas, the s -wave scattering is forbidden due to the Pauli exclusion principle and thus the p -wave interaction is dominant. Furthermore, the

p -wave interaction can be greatly enhanced by the Feshbach resonances [18, 19, 20, 21] and using a p -wave Feshbach resonance between ^{40}K atoms Jin's group at JILA have successfully produced and detected molecules with lifetimes on the order of milliseconds on both the BEC and the BCS side of the resonance [22]. For a 1D gas, the additional confinement induced resonance permits one to tune the 1D effective interaction via a 3D Feshbach resonance [5, 6, 21].

In this paper, we report on a detailed study of the 1D Fermi gases in the infinitely deep square potential well. We will show that the model of fermionic gases with attractive p -wave interactions in such a one-dimensional hard-wall trap is exactly solvable by the Bethe-ansatz method. The experimental efforts in trapping ultracold gases near micro-fabricated surfaces, the so-called "atom chips" [23, 24], and various innovative features in designing the optical box trap [25], are specifically aimed at studying the surprisingly rich variety of physical regimes predicted for the 1D Bose gas and have stimulated many theoretical studies on the physics in a box trap [26, 27, 28]. Different from the harmonic trap, the interacting model in a hard-wall trap is integrable and thus could provide us some exact pictures for understanding the trapped many-body systems. So far, there has been a growing interest in the exactly solved models in the hard-wall trap [26, 29, 30], but most of them focus on the Bose gas and the Fermi gas with odd-wave interactions is not addressed. While the theoretical understanding of the correlation effect of bosonic system has been investigated extensively [5, 7, 8, 9, 10, 11], the fermionic system is not well understood except in the so-called FTG limit [14, 15, 16, 17, 31].

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II. FORMULATION OF THE MODEL AND ITS EXACT SOLUTION

We consider an N -particle system with finite, attractive p -wave interaction in a one-dimensional box of length L , which obviously fills the gap between the FTG limit and free Fermions. The Schrödinger equation can be formulated as

$$\left[-\sum_{i=1}^N \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} + \sum_{1 \leq i < j \leq N} V(x_i - x_j) \right] \Psi = E\Psi, \quad (1)$$

where $V(x_i - x_j)$ is the pseudo-potential describing the p -wave scattering. It has been shown that the p -wave scattering of two spin-polarized fermions in a tightly confined waveguide can be well described by the contact condition [6, 14]

$$\begin{aligned} \Psi_F(x_i - x_j = 0^+) &= -\Psi_F(x_i - x_j = 0^-) \\ &= -a_{1D}^F \frac{\partial}{\partial x} \Psi_F(x_i = x_j \pm 0), \end{aligned} \quad (2)$$

where

$$a_{1D}^F = \frac{3a_p^3}{l_\perp^2} \left[1 + \frac{3\zeta(3/2)}{2\sqrt{2}\pi} \left(\frac{a_p}{l_\perp} \right)^3 \right]^{-1} \quad (3)$$

is the effective 1D scattering length with a_p the p -wave scattering length and $l_\perp = \sqrt{\hbar/m\omega_\perp}$ the transverse oscillator length [14]. The contact condition can be reproduced by using the following pseudo-potential [14, 32]

$$V(x) = -\frac{2\hbar^2 a_{1D}^F}{m} \frac{\partial}{\partial x} \delta(x) \frac{\partial}{\partial x} \quad (4)$$

where $x = x_i - x_j$ and $\partial_x = (\partial_{x_i} - \partial_{x_j})/2$. The scattering length can be tuned readily from 0 to $-\infty$ by sweeping an external magnetic field - the Feshbach resonance, or by changing the geometry of the trapping potential - the confinement induced resonance and in this paper the full physical regimes $-\infty < a_{1D}^F < 0$ will be studied. Similar to the case of Bose gas, the important parameter characterizing the different physical regimes of the 1D Fermi gas is $\gamma = mg_{1D}^F \rho / \hbar^2$, where $g_{1D}^F = -2\hbar^2 a_{1D}^F / m$ and $\rho = N/L$.

A standard rescaling procedure brings the Schrödinger equation into a dimensionless one (for simplicity we keep the original notations)

$$H\Psi(x_1, \dots, x_N) = E\Psi(x_1, \dots, x_N)$$

with

$$H = -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} - 2c_F \sum_{i < j} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \delta(x_i - x_j) \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right),$$

where in the dimensionless interaction constant $2c_F = \sqrt{2m/\hbar^2} a_{1D}^F$ we intentionally keep the factor 2 in accordance with the bosonic case [26]. The wave function takes

the general form

$$\begin{aligned} \Psi(x_1, \dots, x_N) &= \sum_Q \theta(x_{q_N} - x_{q_{N-1}}) \cdots \theta(x_{q_2} - x_{q_1}) \\ &\quad \times \varphi_Q(x_{q_1}, x_{q_2}, \dots, x_{q_N}), \end{aligned} \quad (5)$$

where we have used Q to label the region $0 \leq x_{q_1} \leq x_{q_2} \leq \dots \leq x_{q_N} \leq L$. The wave function of Fermions should follow the antisymmetry of exchange, so our model is simplified into the solution of

$$H\varphi_Q(x_{q_1}, \dots, x_{q_N}) = E\varphi_Q(x_{q_1}, \dots, x_{q_N}) \quad (6)$$

in the region Q with the open boundary condition

$$\varphi_Q(0, x_{q_2}, \dots, x_{q_N}) = \varphi_Q(x_{q_1}, x_{q_2}, \dots, L) = 0.$$

Using the Bethe ansatz method we obtain the wave-function parameterized by the set of quantum number k_1, k_2, \dots, k_N known as quasi-momenta or rapidities [26]

$$\begin{aligned} &\varphi_Q(x_{q_1}, x_{q_2}, \dots, x_{q_N}) \\ &= (-1)^Q \sum_P (-1)^P A_P \exp \left[i \left(\sum_{l < j}^{N-1} \omega_{p_j p_l} \right) \right] \sin(k_{p_1} x_{q_1}) \\ &\quad \times \prod_{1 < j < N} \sin \left(k_{p_j} x_{q_j} - \sum_{l < j} \omega_{p_l p_j} \right) \\ &\quad \times \exp(ik_{p_N} L) \sin(k_{p_N} (L - x_{q_N})) \end{aligned}$$

with $\omega_{ab} = \arctan \frac{1/c_F}{k_a - k_b} - \arctan \frac{1/c_F}{k_b + k_a}$ and $A_{p_1 p_2 \dots p_N} = \prod_{j < l}^N (ik_{p_l} - ik_{p_j} - 1/c_F) (ik_{p_l} + ik_{p_j} - 1/c_F)$. Here $(-1)^Q = \pm 1$ and $(-1)^P = \pm 1$ denote sign factors associated with even/odd permutations of $Q = (q_1, q_2, \dots, q_N)$ and $P = (p_1, p_2, \dots, p_N)$, respectively. The quasi-momenta k_1, k_2, \dots, k_N can be easily determined from the Bethe ansatz equations

$$\exp(2ik_j L) = \prod_{l=1(\neq j)}^N \frac{k_j - k_l - i/c_F}{k_j - k_l + i/c_F} \frac{k_j + k_l - i/c_F}{k_j + k_l + i/c_F}$$

with $j = 1, 2, \dots, N$. These quasi-momenta lead us immediately to important physical quantities for our system. For example, the energy eigenvalue is given by

$$E = \sum_{j=1}^N k_j^2 \quad (7)$$

and the total momentum by $K = \sum_{j=1}^N k_j$. It is clear that these Bethe ansatz equations are the same as those in the case of Bose gas if we simply make a substitution $c = -1/c_F$ [26]. In the regime Q , there is one-to-one correspondence between the quasi-momentum solution of attractive p -wave Fermi gas and that of repulsive Bose gas, but the total wave functions $\Psi(x_1, x_2, \dots, x_N)$ take different forms because of their distinct exchange

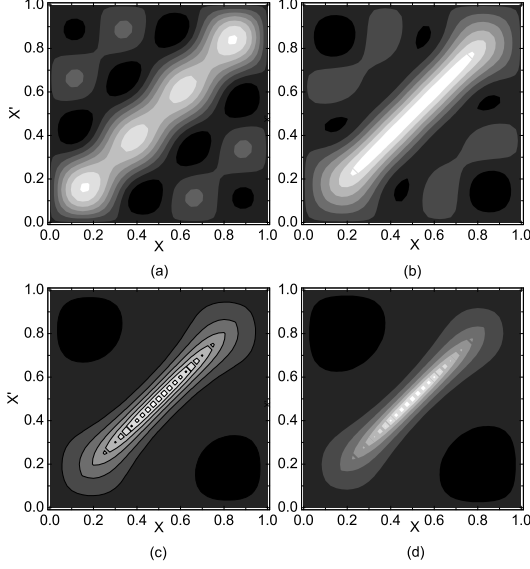


FIG. 1: Gray scale plot of the one-body density matrix of Fermions with p -wave attractive interactions as a function of the dimensionless coordinates x and x' for $N = 4$. (a) $c_F = 0$; (b) $c_F = -0.1$; (c) $c_F = -1$; (d) $c_F = -\infty$.

symmetries. This difference can be easily seen from the one-body density matrix and the momentum distribution. Taking the logarithm of Bethe ansatz equations, we have

$$k_j L = n_j \pi + \sum_{l=1(\neq j)}^N \left(\arctan \frac{1/c_F}{k_l - k_j} - \arctan \frac{1/c_F}{k_j + k_l} \right) \quad (8)$$

For the ground state the set of integer $n_j = 1$ ($1 \leq j \leq N$). For simplicity, in the following evaluation the length L will be taken as one. It is obvious that the solutions of k_j is only relevant to c_F/L for different L . The subsequent procedure is standard. By numerically solving the transcendental equations eqs.(8), we obtain the quasi-momenta k_j and thus the ground state wave function. In principle, all necessary information about the system can be inferred, including the one-body density matrix, the momentum distribution, and the excitation spectrum. Furthermore, one can apply the thermodynamic formalism for dealing with the one dimensional interacting systems developed by Yang and Yang [34].

Before we proceed to the general case with intermediate interaction strength, we'd like to take a look at the situation in the two limiting cases. In the limit of free Fermion case, we have $k_j = j\pi$ ($j = 1, \dots, N$). In this exceptional case of $c_F = 0$, the quasi-momentum acquires the genuine physical meaning of momentum and the atoms occupy the single particle momentum states according to Pauli exclusion principle. We thus have the

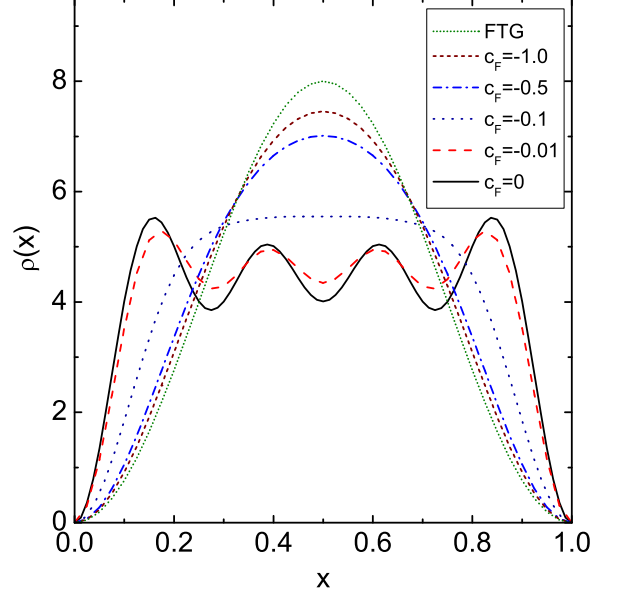


FIG. 2: (color online) Density distribution of Fermions with p -wave attractive interactions for $N = 4$.

total wave function

$$\begin{aligned} \Psi(x_1, x_2, \dots, x_N) &= C \sum_Q (-1)^Q \theta(x_{q_N} - x_{q_{N-1}}) \cdots \theta(x_{q_2} - x_{q_1}) \\ &\times \sum_P (-1)^P \prod_{j=1}^N \sin k_{p_j} x_{q_j}, \end{aligned}$$

with C the normalization constant, which is nothing but the Slater determinant of the lowest N eigenstates $\sin(j\pi x)$ of the system of single particle in the hard wall potential. In the other limit of strongly attractive interaction, i.e., the FTG limit, all quasi-momenta take the same value $k_j = \pi$ ($j = 1, \dots, N$) and $\varphi_Q(x_{q_1}, x_{q_2}, \dots, x_{q_N}) = C(-1)^Q \prod_{j=1}^N \sin \pi x_{q_j}$. Thus we have the total wave function

$$\begin{aligned} \Psi(x_1, x_2, \dots, x_N) &= C \sum_Q (-1)^Q \theta(x_{q_N} - x_{q_{N-1}}) \cdots \theta(x_{q_2} - x_{q_1}) \prod_{j=1}^N \sin \pi x_{q_j}. \end{aligned}$$

On the other hand, the Fermi-Bose mapping method has been used to give exactly the ground state of the FTG gas in [16]

$$\Psi(x_1, x_2, \dots, x_N) = C A(x_1, x_2, \dots, x_N) \prod_{j=1}^N \sin \pi x_j,$$

with the antisymmetric function $A(x_1, x_2, \dots, x_N) = \prod_{1 \leq j < l \leq N} \text{sgn}(x_l - x_j)$. It is obvious that the above two wave functions match each other.

III. GROUND STATE PROPERTIES AND COMPARISON WITH BOSONS

We now turn to the system with finite interaction strength $-\infty < c_F < 0$. In this case the quasi-momenta are decided by numerically solving the Bethe ansatz equations and the total wave function is obtained by Eq.(5) through $\varphi_Q(x_{q_1}, x_{q_2}, \dots, x_{q_N})$ under the restriction of exchange antisymmetry. For the one dimensional interacting system, a quantity of fundamental importance in many-body physics is the one-body density matrix, which, in terms of the ground state wave function $\Psi(x_1, \dots, x_N)$, is given by

$$\rho(x, x') = \frac{N \int_0^L dx_2 \cdots dx_N \Psi^*(x, x_2, \dots, x_N) \Psi(x', x_2, \dots, x_N)}{\int_0^L dx_1 \cdots dx_N |\Psi(x_1, x_2, \dots, x_N)|^2}.$$

This quantity furnishes the expectation values of single particle observables such as the position density distribution $\rho(x) = \rho(x, x')|_{x=x'}$, and the momentum distribution which is simply the Fourier transformation of $\rho(x, x')$,

$$n(k) = \frac{1}{2\pi} \int_0^L dx \int_0^L dx' \rho(x, x') e^{-ik(x-x')}. \quad (9)$$

We display the one-body density matrix and the position density distribution in Fig. 1 and in Fig. 2 for different p -wave attractive interactions. The one-body density matrix expresses the self correlation and it means the probability that two successive measurements, one immediately following the other, will find the particle at the point x and x' , respectively. We notice that for all interacting strengthes there exists a strong enhancement of the diagonal contribution $\rho(x, x')$ along the line $x = x'$. To see this more clearly the position density distribution $\rho(x)$ for $N = 4$ particles is shown in Fig. 2 with more variable interacting strengthes. For noninteracting system the atoms behave as ideal Fermions and the density profiles show obvious spatial oscillation structure. Increasing the strength of the attractive p -wave interaction, as seen in Fig. 2, first leads to the depression of the amplitude of the oscillations of the density profile, followed by the emergence of the typical Gaussian-like bosonic behavior and contraction of the half-width of the density. In the limit of infinitely strong attractive interaction between the Fermions, the system enters into the FTG regime, and the density shows the same smooth profile as that of noninteracting Bosons. Particularly, the density distribution in Fig. 2 for FTG is identical to that of the free Bose gas and there exists one-to-one correspondence for the density distribution between the

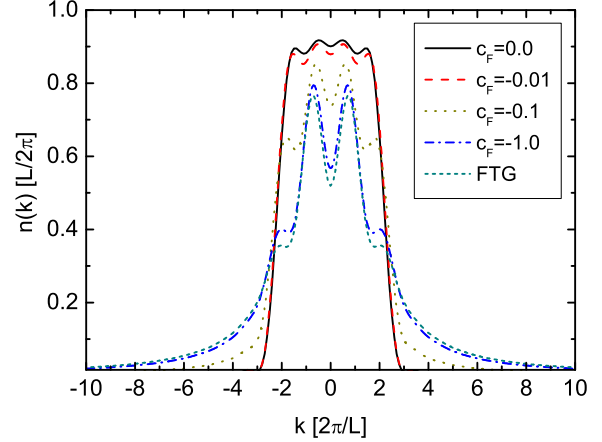


FIG. 3: (color online) Momentum distribution of Fermions with p -wave attractive interactions for $N = 4$.

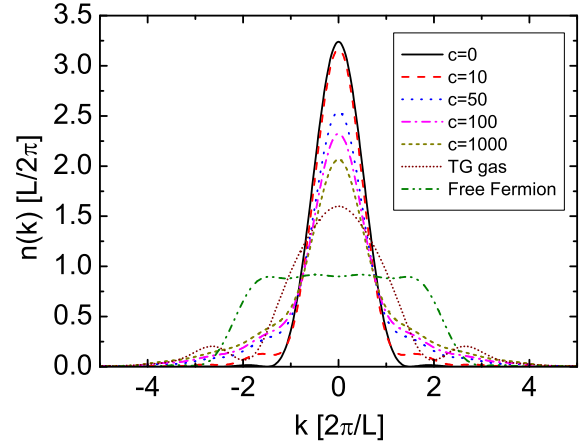


FIG. 4: (color online) Momentum distribution of Bosons with repulsive interactions for $N = 4$.

attractive fermionic gas and the repulsive bosonic gas with the interacting strength related by $c_F = -1/c$ [26].

It is worth to emphasize that the density distribution is identical for TG bosons and noninteracting fermions, and also for FTG fermions and noninteracting bosons. Nevertheless, their momentum distributions show remarkable differences as shown in Fig. 3 for fermions with different attractive interaction strengths and in Fig. 4 for bosons with the corresponding repulsive interaction strengths. The momentum distribution for fermions oscillate in the full regime and the number of oscillation peaks remains equal to the number of atoms in the system. It becomes more and more nonuniform in the momentum space and two sharp spikes indicating that there is high prob-

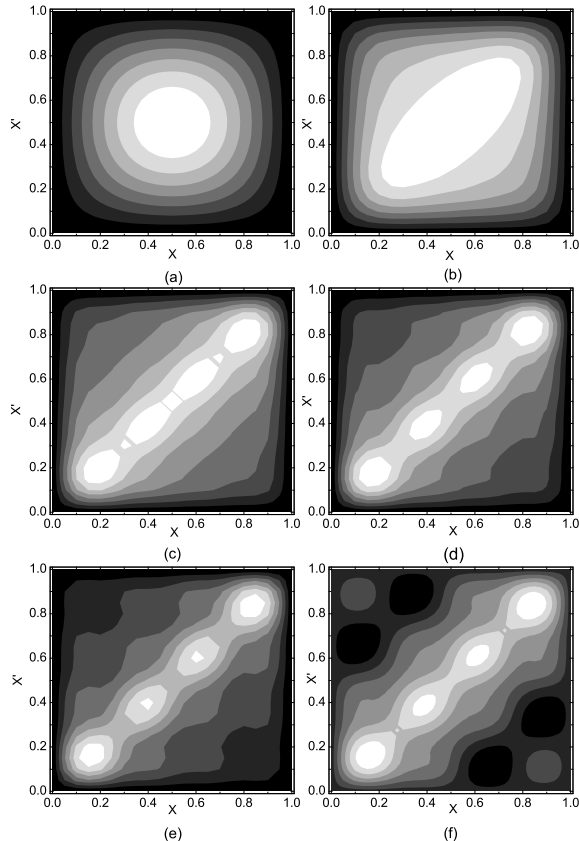


FIG. 5: Gray scale plot of the one-body density matrix of Bosons with repulsive interactions for $N = 4$. (a) $c = 0$; (b) $c = 10$; (c) $c = 50$; (d) $c = 100$; (e) $c = 1000$; (f) $c = \infty$.

ability of finding the atom in momentum states around $k \sim 1$. Two other peaks at higher momentum diminish with increasing interaction but remain there even in the FTG limit due to the fermionic statistics. The half-width of the profiles become larger as the attractive interaction increases.

As a comparison, the momentum distributions of Bosons with repulsive interactions are given in Fig. 4 which are obtained by the Fourier transformation of the one-body density matrix shown in Fig. 5. Obviously, the momentum distribution for the bosons exhibits quite different behaviors. As shown in Fig. 4, there is an obvious peak around the zero momentum point and the height of the peak shrinks with the increase of the repulsive interaction. Furthermore we find no oscillation at all in the momentum distribution for bosons. Even in the Tonks limit, the momentum distribution of Bosons does not show shell structure like free Fermions. The largest probability of distribution appears around the zero point of momentum and decreases rapidly for finite momentum values. Stronger repulsive interaction between the bosonic atoms tends to spread out the distribution into

higher momentum space. Although the momentum distribution for the interacting bosonic gases has been studied by different numerical methods [35, 36], an exact result has never been given except in the TG limit [37, 38].

From the eqs. (7) and (8), we see that the energy level structure of our fermionic model is exactly the same as the corresponding bosonic model with interaction strength related by $c = -1/c_F$. Therefore the thermodynamic properties of the fermionic atoms with attractive p -wave interactions are the same as the well known thermodynamic properties of the 1D boson gas [29, 33, 34] with inverse coupling $c = -1/c_F$. This implies that there exists a Bose-Fermi duality between the p -wave fermionic model and bosonic one with arbitrary interactions which can not be distinguished by the thermodynamic properties. However, due to the different exchange symmetry of the wave functions, the observables associated with the wave functions rather than the square of wave functions (density distribution) should display different behaviors. As we have shown, the off-diagonal density matrix and the momentum distributions are different greatly, which result from the different statistics properties followed by Bosons and Fermions.

IV. CONCLUSIONS

In summary, we have investigated the ground-state properties of fermionic gases with attractive p -wave interactions in a one-dimensional hard-wall trap. With the Bethe ansatz method, the explicit wave function of the ground state and therefore the one-body density and momentum distributions are obtained. It turns out that the density distributions show one-to-one correspondence between Fermions with attractive p -wave interactions and Bosons with repulsive interactions. For weak attractive interaction the density distributions of Fermions display shell structures and the Boson-like distributions appear as the interactions become stronger. In the FTG limit, the Fermi gas exhibits the same distribution as that of the ideal Bose gas. This again confirms the Bose-Fermi duality: strongly interacting Bosons behave like Fermions, and vice versa. Nevertheless, from the viewpoint of momentum distribution, the conclusion is rather different. In the full interacting regime the momentum distributions of Fermions show typical oscillations, which is in sharp contrast with the case of bosonic atoms. In the Tonks limit of infinite interaction, although the density profiles of one dimensional bosons display the Fermion-like distribution, the momentum distribution is still Boson-like.

Acknowledgments

S.C. is supported by NSF of China under Grant No. 10574150 and programs of Chinese Academy of Sciences.

Y.Z. is supported by 973 Program under Grant No. 2006CB921102 and Shanxi Province Youth Science Foun-

dation under Grant No. 20051001. The authors would like to acknowledge the hospitality of KITPC in Beijing.

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